

Small growth vectors of the compactifications of the contact systems on $J^r(1, 1)$

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Abstract

It is well known that the compactifications of the canonical contact systems living on real jet spaces $J^r(1, 1)$, $r \geq 2$, are locally universal Goursat distributions, Δ^r , living on compact manifolds (called Goursat monsters) having open dense jet-like ($J^r(1, 1)$ -like) parts.

By virtue of the results of Jean (1996), one was able, for each $r \geq 2$, to recursively compute the *small growth vector* of Δ^r at any point of the r -th monster. The result was got by performing series of r operations taken, in function of the local geometry of Δ^r in question, from the set of fixed recursive rules $\{1, 2, 3\}$ (called in the present text S, T, G, respectively). By the local universality of Δ^r one was thus able to compute *all* small growth vectors of all existing Goursat distributions.

In the work of Mormul (2004) proposed were explicit solutions to the series of Jean's recurrences. The solutions uncovered a surprisingly involved underlying arithmetics – a kind of Gödel-like encoding of words over a three-letter alphabet $\{G, S, T\}$ by neat sequences of positive integers. Yet, those formulas, though characterizing really existing objects, appeared as if from thin air, and proofs were postponed to another publication.

In the present contribution we submit proofs of our formulas from 2004. It is not, however, a plain check that our candidates satisfy Jean's recurrences. Under the way we retrieve and re-produce those surprising formulas.

Key words and phrases. Goursat flag, small growth vector, solution of Jean's recurrences, encoding of letter words by sequences of integers

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1 Goursat distributions and their small growth vectors

Goursat flags are certain special nested sequences, in general of variable length r ($2 \leq r \leq n - 2$) of subbundles in the tangent bundle TM to a smooth or analytic n -dimensional manifold M : $D^r \subset D^{r-1} \subset \dots \subset D^1 \subset D^0 = TM$. Namely, one demands, for $l = r, r - 1, \dots, 1$, that (a) the Lie square of D^l be D^{l-1} and (b) $\text{cork } D^l = l$. In other words, that the *big growth vector* of the distribution D^r be $[n - r, n - r + 1, n - r + 2, \dots, n - 1, n]$ at each point of M .

This, very restricted, class of objects was being investigated (intermittently) over the last 120 years, with contributions, among others, by E. von Weber [7] and E. Cartan [1]. They independently proved that every corank- r Goursat distribution D^r on M around a generic point locally behaves in a unique way visualised by the *chained model* – the germ at $0 \in \mathbb{R}^n(x^1, \dots, x^{r+2}; x^{r+3}, \dots, x^n)$ of

$$(\partial_n, \dots, \partial_{r+3}; \partial_{r+2}, \partial_1 + x^3 \partial_2 + x^4 \partial_3 + \dots + x^{r+2} \partial_{r+1}) \quad (\text{A})$$

(these are vector fields generators; effectively used are only first $r + 2$ coordinates). Chained models can be viewed as the simplest instance of a family of local writings (preliminary normal forms) of Goursat distributions, obtained much later by Kumpera and Ruiz. As a matter of fact, Kumpera and Ruiz discovered, to a big surprise of the mathematical community, *singularities* hidden in flags, and their pseudo-normal forms were merely a by-product – visualisations of those singularities.

When $n = r + 2$, a Goursat distribution D^r on M has around generic points a unique local description as the germ at $0 \in \mathbb{R}^{r+2}(t, x, x_1, \dots, x_r)$ of

$$(\partial/\partial x_r, \partial/\partial t + x_1 \partial/\partial x + x_2 \partial/\partial x_1 + \dots + x_r \partial/\partial x_{r-1}). \quad (\text{B})$$

That is, of the canonical contact system on $J^r(1, 1)$. (The independent variable is t , the dependent one is x , and x_1, x_2, \dots, x_r can be viewed as the consecutive derivatives of x with respect to t .) This is precisely the result obtained independently by von Weber and Cartan.

For a theory of Goursat flags it has been essential to have a flexible length r that could be increased without changing a manifold, allowing for flags to be prolonged in length on one and the same manifold. In what follows, however, we deal only with the most classical situation $n = r + 2$, when the initial Goursat distribution D^r is of rank 2 and its flag is already of maximal length $r = n - 2$. For, as long as Goursat distributions are considered locally, this causes no loss of generality. In fact, when $n > r + 2$, in a Goursat D^r locally there always splits off an integrable subdistribution of rank $n - r - 2$ which leaves a second direct summand (of rank 2) inside D^r invariant. Like $(\partial_n, \dots, \partial_{r+3})$ in (A) above, which leaves invariant the summand $(\partial_{r+2}, \partial_1 + x^3 \partial_2 + x^4 \partial_3 + \dots + x^{r+2} \partial_{r+1})$. Compare, for instance, Corollary 1.3 in [6].

Note that the rank-2 and corank- r Goursat distributions have the big growth vector $[2, 3, 4, \dots, r + 1, r + 2]$ at each point.

Definition 1. Small growth vector (sgrv in all what follows) $\{n_j(p)\}$ of a distribution D at a point $p \in M$ is the sequence of dimensions at p of the (local) modules of vector fields V_j , $V_1 = D$, $V_{j+1} = V_j + [D, V_j]$,

$$n_j(p) = \dim V_j(p), \quad j = 1, 2, 3, \dots$$

We are interested only in distributions for which this sequence, for every $p \in M$, attains (sooner or later) the value $\dim M$, with ‘sooner or later’ underscored. Such distributions are called *completely nonholonomic*. When, for such a distribution D , there happens $n_{l-1}(p) < n_l(p) = \dim M$, then l is called the *nonholonomy degree* of D at p .

It is a short Lie algebra exercise to show that the Goursat distributions are completely nonholonomic. Around jet-like points p where the visualisation (B) is effective, there holds $n_j(p) = 1 + j$ for $j = 1, 2, \dots, n - 1$, and then the nonholonomy degree, minimal possible for Goursat in dimension n , is $n - 1 = r + 1$, the number r being the order of jets in the visualisation. For non-jet-like points, even *much* slower sgrv’s (and hence bigger nonholonomy degrees) are possible, as is invoiced in [6] and will be eventually clear from the present work. The biggest nonholonomy degree in dimension n will turn out to be F_n , the n -th Fibonacci number. It is about $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$, compare Remark 2 below for more on that.

A given Goursat distribution D^r may not feature all possible singularities in corank r , nor all possible sgrv’s of Goursat distributions of that corank. But in each corank r there do exist manifolds with Goursat structures of corank r on them that are locally universal – feature all possible singularities, hence also all possible sgrv’s of Goursat distributions of corank r . Such are, for instance, the kinematical models extensively used and analysed in [3], and especially *monster Goursat manifolds* $\mathbb{P}^r(\mathbb{R}^2)$, with locally universal Goursat structures Δ^r living on them, constructed in [4].¹

A natural question reads what are all occurring sgrv’s of Goursat distributions of a fixed corank $r \geq 2$. Or, which is the same for that fixed r , what are all sgrv’s of the distribution Δ^r living on $\mathbb{P}^r(\mathbb{R}^2)$.

We recapitulate here the information necessary for the present contribution. First of all, the germs of Goursat flags of length r are partitioned into F_{2r-3} invariant *geometric classes* labelled (encoded) by words of length r over the alphabet $\{G, S, T\}$ starting with GG and such that GT is not allowed.² We call such words *admissible*. The roots of this concept can be traced back to the paper [3], where the prototypes of geometric classes were called *regions*. By virtue of Jean’s results [3], the sgrv of a Goursat germ depends solely on its geometric class. Hence, in length r there are not more than F_{2r-3} different sgrv’s of Goursat germs.

¹ Goursat monsters could be alternatively built over, say, S^2 or T^2 or yet another compact 2-dimensional manifold, instead of \mathbb{R}^2 . But they would serve just the same objective of local universality, while it is much easier to work with $\mathbb{P}^r(\mathbb{R}^2)$, notwithstanding \mathbb{R}^2 is not compact.

² concerning the count of such words, yielding F_{2r-3} , see Observation 1.10 in [6]

Remark 1. It was not explained in [3] whether the small growth vectors attached to the geometric classes, in any fixed length, were all different. It had remained a fine point for quite a time, having become completely clear only on the solutions' side of Jean's recurrences. See in this respect in [6]: the discussion in Section 2.1 and Theorem 3.5.

Those who prefer the universal objects may think of the geometric classes in length r as just the strata of a [very regular, besides] stratification of the r -th monster $\mathbb{P}^r(\mathbb{R}^2)$. Those strata are embedded submanifolds of $\mathbb{P}^r(\mathbb{R}^2)$ of codimensions equal to the numbers of non-G letters in their codes. (As a matter of fact, Jean's stratification of $\mathbb{P}^r(\mathbb{R})^2$ is labelled in [4] by r -letter words over a newer alphabet $\{R, V, T\}$, with R replacing the previous G, V replacing S, and T replacing T.) Repeating, then, within each stratum, or geometric class, the sgrv of Δ^r is constant, not depending on points.

In fact, it is a nondecreasing sequence of integers that starts with 2 and takes on, for the Goursat distributions, all intermediate integer values up to $r + 2$ inclusively. The key issue is to ascertain with what multiplicities do these integers appear. We recall that the sequence of multiplicities of integers in the sgrv of a given geometric class has been called in [6] the **derived vector** of that class. We stick to this terminology in the present paper. Sometimes, for brevity's sake, we will speak about the derived vector of an admissible word (= the code of a geometric class).

The recipes to compute the derived vectors out of admissible (G,S,T) words which encode the geometric classes, were given in [6]: firstly on a recursive basis rephrasing the results of [3], then in closed form formulas, which explicitly solved the recurrences for derived vectors. The first way did not need a proof, because Jean's recurrences, for the functions *beta* he used, automatically implied the recurrences for derived vectors being nothing but the sequences of *increments* of the beta functions.

The second way necessitated a proof, or proofs of three separate statements in [6]: Proposition 3.2, Theorem 3.3, and Theorem 3.4. The purpose of the present text is to furnish such proofs, after a slight reformatting of the statements under consideration: a part of old Proposition 3.2 and Theorem 3.3 are now merged into Theorem 2, while the rest of Proposition 3.2 and Theorem 3.4 are merged into Theorem 3.

In [6], for each geometric class \mathcal{C} of length r , the derived vector of \mathcal{C} , denoted by $\text{der}(\mathcal{C})$, is a function $\{2, 3, \dots, r + 1\} \rightarrow \mathbb{N}$, in which $\text{der}(\mathcal{C})(j)$ is the multiplicity of j in the sgrv of \mathcal{C} , for every integer $2 \leq j \leq r + 1$. Eventually the sgrv ends with the ceiling value $r + 2$.

So how does Jean recursively arrive (modulo the mentioned passing to the increments of his original functions beta) at the functions $\text{der}(\mathcal{C})$?

Theorem 1 ([3]) *For each geometric class \mathcal{C} in length r , the function $\text{der}(\mathcal{C})$ equals the last term d^r in the sequence of functions d^1, d^2, \dots, d^r that are constructed, by means of operations G, S, T defined below, as follows.*

One defines, recursively for $j = 1, 2, \dots, r$, functions $d^j: \{2, \dots, j+1\} \rightarrow \mathbb{N}$. One starts by declaring $d^1 = (1)$ and $d^2 = (1, 1)$, and then continues

$$d^{j+2} = \begin{cases} G(d^{j+1}), & \text{when the } (j+2)\text{-nd letter in the code of } \mathcal{C} \text{ is G,} \\ S(d^j, d^{j+1}), & \text{when the } (j+2)\text{-nd letter in the code of } \mathcal{C} \text{ is S,} \\ T(d^j, d^{j+1}), & \text{when the } (j+2)\text{-nd letter in the code of } \mathcal{C} \text{ is T.} \end{cases}$$

Concerning the operations in use, the simplest among them is the operation G,

$$G(\alpha) = \begin{pmatrix} 2 & 3 & 4 & 5 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ 1 & \alpha(2) & \alpha(3) & \alpha(4) & \dots \end{pmatrix}.$$

So it is an "insert a 1 on the left and shift an argument vector α by one slot to the right" operation. Clearly, $G(\alpha)$ is a sequence (function) by one entry longer than α (with the domain by one element bigger than the domain of α).

The remaining operations S and T are more involved. They are two-argument, not just one-argument like G. Their first argument, say α , is a sequence by one entry shorter than the second argument, say β , and their outputs are yet by one entry longer than β ,

$$S(\alpha, \beta) = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ 1 & 1 & \alpha(2) + \beta(3) & \alpha(3) + \beta(4) & \alpha(4) + \beta(5) & \dots \end{pmatrix},$$

$$T(\alpha, \beta) = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ 1 & 1 & 2\beta(3) - \alpha(2) & 2\beta(4) - \alpha(3) & 2\beta(5) - \alpha(4) & \dots \end{pmatrix}.$$

So, in perhaps more catching terms, S is a *Fibonacci-like rule*, with appropriate shifts of the input vectors α and β , while T is an *arithmetic progression rule*, with likewise shifts of the inputs. In fact, when the two inputs α and β are identically indented on the left, then their S (T, resp.) output starts, one row below, by inserting two 1's on the left and then performing the Fibonacci rule (arithmetic progression rule, resp.) on α and β skew-wise from the NW to the SE. To give an instance of each of these two-argument operations in action, and skipping brackets for bigger transparency,

$$\begin{array}{rcl} \alpha & = & 1 \quad 1 \quad 2 \quad 3 \\ \beta & = & 1 \quad 1 \quad 1 \quad 2 \quad 3 \\ S(\alpha, \beta) & = & \mathbf{1} \quad \mathbf{1} \quad 2 \quad 2 \quad 4 \quad 6 \end{array}$$

and

$$\begin{array}{rcl} \alpha & = & 1 \quad 1 \quad 2 \\ \beta & = & 1 \quad 1 \quad 2 \quad 3 \\ T(\alpha, \beta) & = & \mathbf{1} \quad \mathbf{1} \quad 1 \quad 3 \quad 4 \end{array}$$

(the two initial 1's inserted in the outputs on the left are written in bold).

Remark 2. When, for any fixed length r , does the sum of values of $\text{der}(\mathcal{C})$ attain the maximal value, or: when the nonholonomy degree $1 + \sum_{i=2}^{r+1} \text{der}(\mathcal{C})(i) = \max$? It is an exercise in the G, S, and T operations to show that, for each r , the maximal nonholonomy degree is realized only by the [most singular] class $\mathcal{C} = \text{GGS}\underbrace{\text{SS}\dots\text{S}}_{r-2}$. Then $\text{der}(\mathcal{C})(i) = F_{i-1}$ for $i = 2, 3, \dots, r+1$, and

$$1 + \sum_{i=2}^{r+1} \text{der}(\mathcal{C})(i) = 1 + F_1 + F_2 + \dots + F_r = F_{r+2},$$

as invoiced earlier in this section. (Thus the nonholonomy degree of the distribution Δ^r always varies between $r+1$ and F_{r+2} and these bounds are sharp for each $r \geq 2$. For instance, for Δ^3 , it varies between 4 and 5. In general, however, *not* all intermediate integer values are realized by the nonholonomy degrees of Δ^r . The smallest length with this happening is 6: the value 20 located between 7 and $21 = F_{6+2}$ is not realized, cf. Theorem 23 in [2].)

1.1 Useful aggregates of geometric classes.

In what follows, the length (r) of a geometric class will not be essential. Much more important will be the number, say $s+1 \geq 1$, of letters S in class' code that can otherwise be arbitrarily long.³ To word our results, we arbitrarily **fix this discrete parameter** $s \geq 0$.

We have to introduce some more, still necessary, parameters – the numbers of letters T and G in classes' words going past the letters S, those letters S being run *backwards* from code's right end to left.

Namely, agree that the last S in a code is followed by $k_0 \geq 0$ letters T, and then by $l_0 \geq 0$ letters G. Agree also that the one before last S is followed by $k_1 \geq 0$ letters T and then by $l_1 \geq 0$ letters G, and so on backwards until the first appearing letter S being followed by $k_s \geq 0$ letters T, then by $l_s \geq 0$ letters G.

On top of that, let a code start with $l_{s+1} \geq 2$ letters G. (Except for this last quantity, all introduced integer parameters may even vanish, as it happens in the 'Fibonacci' classes GGS_{s+1} discussed in Remark 2 above.)

Remark 3. The indexation of integer parameters k_j and l_j proposed here slightly differs, by a backward shift of indices by one, from the indexation in [6]. The same applies to the, central for the contribution, family (depending on the parameters k_j) of sequences (A_0, A_1, A_2, \dots) introduced below before Theorem 2. The formulas, especially in Theorem 3 below, assume thus a more compact view than in [6].

³ By Theorem 1, in any fixed length $r \geq 2$, the generic jet-like geometric class with no letters S, written shortly G_r , and only this class, has the simplest derived vector $(1, 1, \dots, 1)$ (r 1's). Within G_r , and only within it (in that length), the small growth vector of a Goursat distribution is the same as its big growth vector. In the paper we only consider the *remaining* geometric classes featuring certain letters S in their codes. That is, we consider the singular geometric classes.

In terms of these parameters, what can be said about the derived vectors emerging from Theorem 1?

We will first answer the question for the geometric classes having $l_1 = l_2 = \dots = l_s = 0$. That is, for classes in whose codes the letters G occur only in the beginning (in number $l_{s+1} \geq 2$) and possibly in the end (in number $l_0 \geq 0$). The union of such geometric classes has been called in [6] the hyperclass 0_s . In the present text we use the concept of a hyperclass only implicitly, not explicitly.

To word the relevant theorem, let us introduce a *family* of integer sequences

$$\bullet \quad A_0 = 1, \quad A_1 = 2 + k_0, \quad A_j = A_{j-2} + (1 + k_{j-1})A_{j-1} \quad \text{for } 2 \leq j \leq s+1.$$

(This family is parametrized by the non-negative integers k_0, k_1, \dots, k_s .)

2 Main theorems

Theorem 2 ([6]) *In the derived vectors of the geometric classes with $s+1$ letters S and such that $l_1 = l_2 = \dots = l_s = 0$, there appear the following $s+2$ different values, in the growing order: A_0, A_1, \dots, A_{s+1} . All these derived vectors are non-decreasing and, concerning the multiplicities of the listed values:*

- A_0 appears $2 + k_0 + l_0$ times in row;
- for $1 \leq j \leq s$, the number A_j appears $1 + k_j$ times in row;
- A_{s+1} appears $l_{s+1} - 1$ times in row.

Proof of this theorem is given in Section 3.

Before describing the sgrv's of the remaining geometric classes, we need an infinite series of sequences of the type A . This series will be parametrized by yet another natural number N . (N is subject to some restrictions in function of s , but s is not à priori bounded from above.)

- $A_0^{+N} = 1, \quad A_1^{+N} = 2 + k_N \quad (1 \leq N \leq s),$
- $A_j^{+N} = A_{j-2}^{+N} + (1 + k_{j-1+N})A_{j-1}^{+N} \quad \text{for } 2 \leq j, \quad N + j \leq s+1.$

We also have to specifically put in relief those letters S in a class' code that are followed by **non-zero numbers** of letters G (going past some letters T, if any). They will play a central role in the characterization of the derived vector of such a class.

Let, for a given geometric class not served by Theorem 2 (hence, in particular, having $s+1 \geq 2$ letters S in its code),

$$1 \leq n_1 < n_2 < \dots < n_q \leq s$$

be *all* indices i between 1 and s such that $l_i > 0$. (The index n_1 exists for each such class. The index n_2 – already not for each such class. For its existence, the

parameter s should be at least 2 and, among the discrete parameters l_1, l_2, \dots, l_s at least two should be non-zero, etc.) With these notations, we are now ready to complete the information given in Theorem 2 by the following

Theorem 3 ([6]) *In the derived vectors of the geometric classes with $s+1$ letters S and such that, among the parameters $\{l_1, l_2, \dots, l_s\}$, exactly $l_{n_1}, l_{n_2}, \dots, l_{n_q}$ are non-zero (positive), $1 \leq n_1 < n_2 < \dots < n_q \leq s$, $1 \leq q \leq s$, there appear $s+2$ different values, listed below in the growing order, in the following $q+1$ separate rows:*

$$\begin{aligned}
& A_0, A_1, \dots, A_{n_1-1}; \\
& A_{n_1}(A_0^{+n_1}, A_1^{+n_1}, \dots, A_{n_2-n_1-1}^{+n_1}); \\
& A_{n_1}A_{n_2-n_1}^{+n_1}(A_0^{+n_2}, A_1^{+n_2}, \dots, A_{n_3-n_2-1}^{+n_2}); \\
& \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
& A_{n_1} \prod_{j=1}^{q-2} A_{n_{j+1}-n_j}^{+n_j} \left(A_0^{+n_{q-1}}, A_1^{+n_{q-1}}, \dots, A_{n_q-n_{q-1}-1}^{+n_{q-1}} \right); \\
& A_{n_1} \prod_{j=1}^{q-1} A_{n_{j+1}-n_j}^{+n_j} \left(A_0^{+n_q}, A_1^{+n_q}, \dots, A_{s-n_q+1}^{+n_q} \right).
\end{aligned}$$

These derived vectors are always non-decreasing and, among the above-listed values:

- The 0-th of them (A_0) appears $2 + k_0 + l_0$ times in row;
- for $1 \leq j \leq s$, the j -th value appears $1 + k_j + l_j$ times in row (that is, $1 + k_j$ times, when $j \notin \{n_1, n_2, \dots, n_q\}$);
- the last and biggest $(s+1)$ -st value appears $l_{s+1} - 1$ times in row.

Observations. (i) The table of $s+2$ different values in $\text{der}(\mathcal{C})$ in Theorem 3 is now formatted slightly differently than in Theorem 3.4 in [6]. The last entries in the rows (except the last row) *there are now* shifted to the front of the following rows. This is done by purely technical reasons of making the proof (in Section 4) more transparent.

(ii) Also, to simplify the presentation in Theorem 3, the numbers standing *before* the brackets in the all but first rows in the above list do multiply all number entries standing *inside* the relevant brackets.

(iii) When $n_1 = 1$, or when $n_{i+1} - n_i = 1$, then there is only one entry in the relevant row in the list. When $n_q = s$, there are only two entries in the last row.

Proof of Theorem 3 is given in Section 4.

Example 1. We want to illustrate Theorem 3 on one geometric class taken from the list of 93 *orbits* of the local classification of Δ^7 in [5]. (The language of [5] was different, but the objective of that paper was to locally classify Δ^7 .)

Namely, for $\mathcal{C} = \text{GGSTSGS}$ (called 3.1.3.2.3 in [5]; it is a single orbit in $\mathbb{P}^7\mathbb{R}^2$) we have $s = 2$, $q = 1$, $n_1 = 1$ and $k_0 = l_0 = k_1 = 0$, $l_1 = k_2 = 1$, $l_2 = 0$, $l_3 = 2$. By Theorem 3, the table of different values appearing in $\text{der}(\mathcal{C})$ (consisting of $q + 1 = 2$ rows) is

$$\begin{aligned} & A_0; \\ & A_1(A_0^{+1}, A_1^{+1}, A_2^{+1}), \end{aligned}$$

or else, after the due substitutions,

$$\begin{aligned} & 1; \\ & 2(1, 2, 5). \end{aligned}$$

By the same Theorem 3, the multiplicities of these values are $2 + k_0 + l_0 = 2$, $1 + k_1 + l_1 = 2$, $1 + k_2 + l_2 = 2$, $l_3 - 1 = 1$, respectively. Hence $\text{der}(\mathcal{C}) = (1, 1, 2, 2, 4, 4, 10)$.

Let us look now how Jean's recurrences lead to the same result:

$$\begin{array}{rcccccc} \text{G} & 1 & & & & & \\ \text{G} & 1 & 1 & & & & \\ \text{S} & 1 & 1 & 2 & & & \\ \text{T} & 1 & 1 & 1 & 3 & & \\ \text{S} & 1 & 1 & 2 & 2 & 5 & \\ \text{G} & 1 & 1 & 1 & 2 & 2 & 5 \\ \text{S} & 1 & 1 & 2 & 2 & 4 & 4 & 10 \end{array}$$

(omitting the separating commas in the vectors).

Remark 4. In [6] there is Remark 4.1 which says that, in the end of the day, the two-step recurrences from [3] recalled in Theorem 1, can be replaced by [much less transparent, however] one-step recurrences. We want to underline that it is but an immediate consequence of Theorems 2 and 3, for the derived vectors emerging from these theorems are just *functions* of the codes of geometric classes under consideration. Functions of the extensible set of arguments $s, k_0, l_0, k_1, l_1, \dots, k_s, l_s, l_{s+1}$.

To make a point on these one-step recurrences that are now virtually at hand:

- Prolonging a geometric class by G (increasing l_0 by 1) results in an extremely simple operation on the derived vector.
- Prolonging it by T (meaning that $l_0 = 0$) is just increasing k_0 by 1, with the resulting changes in the derived vector.
- Endly, prolonging it by S means: -increasing the parameter s by 1, - the related shift of indices in the parameters k and l , - the consequent changes in the derived vector, all of them governed by Theorems 2 and/or 3.

3 Proof of Theorem 2

The proof is by induction on $s \geq 0$. For $s = 0$, i.e., for the geometric classes with just one letter S in their codes ($s + 1 = 1$), the derived vectors can be computed directly from Theorem 1. Indeed, for the classes $G_{l_1} S T_{k_0} G_{l_0}$, where $l_1 \geq 2$, we are to perform $l_1 - 2 \geq 0$ operations G starting from the function d^2 , then one operation S, then $k_0 \geq 0$ operations T, and eventually $l_0 \geq 0$ operations G.

We know from the beginning that $d^1 = (1)$, $d^2 = (1, 1)$. So, after $l_1 - 2$ operations G,

$$d^{l_1} = (\underbrace{1, 1, \dots, 1}_{l_1}) = (1_{l_1}).$$

Now, by the Fibonacci-like rule in the operation S,

$$d^{l_1+1} = S(d^{l_1-1}, d^{l_1}) = (1, 1, 2_{l_1-1}). \quad (1)$$

When $k_0 > 0$, we next perform k_0 operations T, that is, compute k_0 subsequent terms in arithmetic progressions with the two initial terms either $(1, 1)$ or $(1, 2)$:

$$\begin{aligned} d^{l_1+2} &= T(d^{l_1}, d^{l_1+1}) = (1, 1, 1, 3_{l_1-1}) \\ &\dots \quad \dots \quad \dots \quad \dots \\ d^{l_1+1+k_0} &= (1_{2+k_0}, (2+k_0)_{l_1-1}). \end{aligned} \quad (2)$$

Observe that, when $k_0 = 0$, the expression on the right in (1) coincides with the expression on the right in (2). So (2) is also valid for $k_0 = 0$. Now, applying the operation G l_0 times to the sequence on the RHS of (2), we get

$$d^{l_1+1+k_0+l_0} = (1_{2+k_0+l_0}, (2+k_0)_{l_1-1}) = (\underbrace{A_0, \dots, A_0}_{2+k_0+l_0}, \underbrace{A_1, \dots, A_1}_{l_1-1}),$$

as stated in theorem's statement, irrespectively of whether $l_0 > 0$ or $l_0 = 0$. The beginning of induction is done.

We are now to justify, for any fixed integer value $s \geq 0$, the induction step ' $s \Rightarrow s + 1$ '. That is, we assume theorem's statement for classes with **not more than** $s + 1$ letters S in their codes and work towards the analogous statement for an arbitrary class \mathcal{C} having $s + 2$ letters S and with such discrete parameters that $l_1 = l_2 = \dots = l_{s+1} = 0$.

It is natural to truncate the code of \mathcal{C} just before the last from the left (the first from the right!) letter S in it. The inductive hypothesis will be applied to this truncated word, with the up-shift of indices in theorem's parameters by 1 and – attention – with the last parameter l_0 in the very wording of theorem (its role played now by l_1) vanishing. In parallel, the hypothesis will also be applied to this truncated word with *its* last (or: first from the right) letter *deleted*. We are going to treat their derived vectors as known and explicitly described by the theorem under proof.

Observe that the latter (doubly truncated) word, may have either $s + 1$ or s letters S. The first case occurs, clearly, when, there is a gap in the code of \mathcal{C} between its $(s + 1)$ -st and $(s + 2)$ -nd letter S counted from the left (between the first and second letter S counted from the right). That is, when $k_1 \geq 1$. The second case – when there is no mentioned gap, or else when $k_1 = 0$. We will argue completely separately in each case, for the information concerning the derived vector of the doubly truncated word depends critically on the case occurring.

3.1 Case $k_1 \geq 1$.

In this case, the above-mentioned truncated words are of the form \mathbf{WST}_{k_1} and \mathbf{WST}_{k_1-1} , where \mathbf{W} is an admissible word with s letters S.

So, by the inductive assumption, the vector $\text{der}(\mathbf{WST}_{k_1})$ is non-decreasing and features (with due multiplicities precised in the inductive assumption) the numbers

$$A_0^{+1}, A_1^{+1}, \dots, A_s^{+1}, A_{s+1}^{+1},$$

while the vector $\text{der}(\mathbf{WST}_{k_1-1})$ is also non-decreasing and features formally ‘similar’ numbers (with their proper multiplicities also precised in the inductive assumption), yet built on the basis of the initial data $k_1 - 1, k_2, \dots, k_{s+1}$ instead of k_1, k_2, \dots, k_{s+1} in the previous case. Let us write those numbers down as

$$\bar{A}_0^{+1}, \bar{A}_1^{+1}, \dots, \bar{A}_s^{+1}, \bar{A}_{s+1}^{+1}.$$

These are our basic construction bricks. With their help we will firstly express the vector

$$S\left(\text{der}(\mathbf{WST}_{k_1-1}), \text{der}(\mathbf{WST}_{k_1})\right). \quad (3)$$

In order to perform this operation S, one writes its vector arguments one above the other, identically indented on the left.

So, in one row, called $(*)$, from the left to right:

- $2 + k_1 - 1 = 1 + k_1$ times \bar{A}_0^{+1} , then
- $1 + k_2$ times \bar{A}_1^{+1} , then
- $\dots \dots \dots$
- $1 + k_{s+1}$ times \bar{A}_s^{+1} , and then
- $l_{s+2} - 1$ times \bar{A}_{s+1}^{+1} .

And then in the following row, called $(**)$, also from the left to right:

- $2 + k_1$ times A_0^{+1} , then
- $1 + k_2$ times A_1^{+1} , then
- $\dots \dots \dots$

- $1 + k_{s+1}$ times A_s^{+1} , and, eventually,
- $l_{s+2} - 1$ times A_{s+1}^{+1} .

In the subsequent row, indented on the left exactly as (*) and (**) are, we ought, by the definition of S, to start with two 1's and then perform the Fibonacci rule skew-wards from NW to SE. Let us present this procedure in detail in the initial parts of the row (*) (with its $1 + k_1$ initial entries) and (**) (with its $2 + k_1$ initial entries):

$$\begin{array}{ccccccc}
\bar{A}_0^{+1} & \dots & \bar{A}_0^{+1} & \bar{A}_0^{+1} & & & \\
A_0^{+1} & A_0^{+1} & \dots & A_0^{+1} & A_0^{+1} & & \\
\mathbf{1} & \mathbf{1} & A_0^{+1} + \bar{A}_0^{+1} & \dots & A_0^{+1} + \bar{A}_0^{+1} & A_0^{+1} + \bar{A}_0^{+1} &
\end{array}$$

Given that we know the entire rows (*) and (**), it is now visible that the overall result of the operation S is a row, called (***), by one entry longer than (**), in which there go, from the left to right,

- 2 times 1, then
- $1 + k_1$ times $A_0^{+1} + \bar{A}_0^{+1}$, then
- $1 + k_2$ times $A_1^{+1} + \bar{A}_1^{+1}$, then
- $\dots \dots \dots$
- $1 + k_{s+1}$ times $A_s^{+1} + \bar{A}_s^{+1}$, and then
- $l_{s+2} - 1$ times $A_{s+1}^{+1} + \bar{A}_{s+1}^{+1}$.

Now we are to perform the operation T certain ($k_0 \geq 0$) number of times. When $k_0 > 0$, we apply it for the first time to the rows (**) and (***) (these two are its vector arguments). Let us present this procedure in detail, like with the preceding operation, in the initial parts of these rows, focusing on the $2 + k_1$ initial entries in the row (**) and on the $3 + k_1$ initial entries in the row (***):

$$\begin{array}{ccccccc}
1 & A_0^{+1} & A_0^{+1} & \dots & A_0^{+1} & & \\
1 & 1 & A_0^{+1} + \bar{A}_0^{+1} & A_0^{+1} + \bar{A}_0^{+1} & \dots & A_0^{+1} + \bar{A}_0^{+1} & \\
\mathbf{1} & \mathbf{1} & 1 & A_0^{+1} + 2\bar{A}_0^{+1} & A_0^{+1} + 2\bar{A}_0^{+1} & \dots & A_0^{+1} + 2\bar{A}_0^{+1}
\end{array}$$

Given that we know the entire rows (**) and (***), it now becomes clear (a) how the rest of the output of this first operation T looks like, and (b) how the entire continuation with the consecutive $k_0 - 1$ operations T looks like. Namely, the resulting eventual derived vector consists of the entries

- $2 + k_0$ times $1 (= A_0)$, then
- $1 + k_1$ times $A_0^{+1} + (1 + k_0)\bar{A}_0^{+1}$, then
- $1 + k_2$ times $A_1^{+1} + (1 + k_0)\bar{A}_1^{+1}$, then

-
- $1 + k_{s+1}$ times $A_s^{+1} + (1 + k_0)\bar{A}_s^{+1}$, and then
- $l_{s+2} - 1$ times $A_{s+1}^{+1} + (1 + k_0)\bar{A}_{s+1}^{+1}$.

Observe that, when $k_0 = 0$ (i.e., when there is no operation T after the S) this result also takes effect, because it coincides with the row $(***)$.

What we still need in this case $k_1 \geq 1$ is

Lemma 1 $A_j^{+1} + (1 + k_0)\bar{A}_j^{+1} = A_{j+1}$ for $j = 0, 1, \dots, s, s+1$.

Proof by induction on j . The beginning of induction, for $j = 0$ and $j = 1$, is immediate. Indeed, $A_0^{+1} + (1 + k_0)\bar{A}_0^{+1} = 1 + (1 + k_0) \cdot 1 = A_1$ and

$$A_1^{+1} + (1 + k_0)\bar{A}_1^{+1} = 2 + k_1 + (1 + k_0)(2 + k_1 - 1) = 1 + (2 + k_0)(1 + k_1) = A_0 + (1 + k_1)A_1 = A_2.$$

The induction step, in the form $(j - 1, j) \Rightarrow j + 1$ for any fixed j , $1 \leq j \leq s$, is only a bit longer. From the definition of the A sequences

$$A_{j+1}^{+1} = A_{j-1}^{+1} + (1 + k_{j+1})A_j^{+1}, \quad (4)$$

$$\bar{A}_{j+1}^{+1} = \bar{A}_{j-1}^{+1} + (1 + k_{j+1})\bar{A}_j^{+1}. \quad (5)$$

Now we check if lemma's statement holds for $j + 1$:

$$\begin{aligned} A_{j+1}^{+1} + (1 + k_0)\bar{A}_{j+1}^{+1} &\stackrel{\text{by (4) and (5)}}{=} A_{j-1}^{+1} + (1 + k_{j+1})A_j^{+1} + (1 + k_0)(\bar{A}_{j-1}^{+1} + (1 + k_{j+1})\bar{A}_j^{+1}) \\ &= A_{j-1}^{+1} + (1 + k_0)\bar{A}_{j-1}^{+1} + (1 + k_{j+1})(A_j^{+1} + (1 + k_0)\bar{A}_j^{+1}) \\ &\stackrel{\text{by ind. ass.}}{=} A_j + (1 + k_{j+1})A_{j+1} = A_{j+2}. \end{aligned}$$

Taking into account this lemma, after the operations S and $k_0 \geq 0$ times T we have the result row in a clearer form

- $2 + k_0$ times A_0 , then
- $1 + k_1$ times A_1 , then
- $1 + k_2$ times A_2 , then
-
- $1 + k_{s+1}$ times A_{s+1} , and then
- $l_{s+2} - 1$ times A_{s+2} .

This is the vector $\text{der}(\mathbf{WST}_{k_1}\mathbf{ST}_{k_0})$. To finish the case $k_1 \geq 1$, we remember that there still are $l_0 \geq 0$ letters G in the end of the code of \mathcal{C} . But the operation G is extremely simple: one just adds, from the left, one entry 1. Hence the eventual vector $\text{der}(\mathbf{WST}_{k_1}\mathbf{ST}_{k_0}\mathbf{G}_{l_0})$ has just l_0 entries 1 ($= A_0$) more in the beginning. That is, it consists of

- $2 + k_0 + l_0$ times A_0 , then
- $1 + k_1$ times A_1 , then
- $1 + k_2$ times A_2 , then
- $\dots \dots \dots$
- $1 + k_{s+1}$ times A_{s+1} , and then
- $l_{s+2} - 1$ times A_{s+2} .

Case $k_1 \geq 1$ [in the inductive proof of Theorem 2] is now proved.

3.2 Case $k_1 = 0$.

Now the two truncated words are just of the form \mathbf{WS} and \mathbf{W} , where \mathbf{W} is an admissible word with s letters S.

The word \mathbf{WS} is admissible, too, and has $s + 1$ letters S. So, by the inductive assumption, the vector $\text{der}(\mathbf{WS})$ is non-decreasing and features, with due multiplicities precised in the inductive assumption, the numbers, written in the growing order:

$$A_0^{+1}, A_1^{+1}, \dots, A_s^{+1}, A_{s+1}^{+1},$$

now with an important restriction $k_1 = 0$, hence $A_1^{+1} = 2 + k_1 = 2$ etc. The word \mathbf{W} having s letters S, by the same assumption, the vector $\text{der}(\mathbf{W})$ is non-decreasing and features the numbers, in the growing order (with their proper multiplicities also precised in the inductive assumption),

$$A_0^{+2}, A_1^{+2}, \dots, A_s^{+2}.$$

These are our construction bricks this time. We first have to compute the derived vector $S(\text{der}(\mathbf{W}), \text{der}(\mathbf{WS}))$. To this end we expand $\text{der}(\mathbf{W})$ in a row, called $(*)'$, consisting of

- $2 + k_2$ times A_0^{+2} , then
- $1 + k_3$ times A_1^{+2} , then
- $\dots \dots \dots$
- $1 + k_{s+1}$ times A_{s-1}^{+2} , and, eventually,
- $l_{s+2} - 1$ times A_s^{+2} .

(Clearly, when $s = 0$, then only the last group is present here and $\text{der}(\mathbf{W})$ consists of $l_2 - 1$ values $A_0^{+2} = 1$.)

And we also expand $\text{der}(\mathbf{WS})$, in a row called $(**')$, only slightly differing from the row $(**)$ in the previous case. The difference resides in the current specification $k_1 = 0$ (which generates, naturally, a different sequence of values A than when $k_1 > 0$):

- 2 times A_0^{+1} , then
- $1 + k_2$ times A_1^{+1} , then
-
- $1 + k_{s+1}$ times A_s^{+1} , and, eventually,
- $l_{s+2} - 1$ times A_{s+1}^{+1} .

The continuation is, basically, already known. The row $(*)'$ is written above row $(**')$, both identically indented on the left, and in the following row below them, called $(***')$ and identically indented on the left, we start with two 1's written in bold to put them in relief, then apply the Fibonacci-like rule from the NW to SE. Let us trace down this procedure in the initial parts of the rows: first $2 + k_2$ terms in $(*)'$ and first $3 + k_2$ terms in $(**')$:

$$\begin{array}{ccccccc}
(*)' & 1 & A_0^{+2} & \dots & A_0^{+2} & & \\
(**') & 1 & 1 & A_1^{+1} & \dots & A_1^{+1} & \\
(***)' & \mathbf{1} & \mathbf{1} & 2 & A_1^{+1} + A_0^{+2} & \dots & A_1^{+1} + A_0^{+2}
\end{array}$$

(the first entry in $(*)'$ is written as 1, and not A_0^{+2}). Given the above-listed multiplicities of different values in the rows $(*)'$ and $(**')$, the pattern observed on the initial terms of both the arguments and result smoothly extends to the entire rows. The outcome row $(***') = S((*)', (**'))$ consists of entries, in the order of their appearing:

- 2 times 1, then
- 1 time 2, then
- $1 + k_2$ times $A_1^{+1} + A_0^{+2}$, then
- $1 + k_3$ times $A_2^{+1} + A_1^{+2}$, then
-
- $1 + k_{s+1}$ times $A_s^{+1} + A_{s-1}^{+2}$, and then
- $l_{s+2} - 1$ times $A_{s+1}^{+1} + A_s^{+2}$.

Now we are to continue with the operation T performed k_0 times. Since T is a (skew-wards NW to SE) arithmetic-progression-like operation which is initially being applied after the operation S acting on the rows: $(*)'$ as the first argument in S and $(**')$ as second argument, the arithmetic progressions obtained in the outcome of k_0 operations T are either constant and equal to 1, or have their *differences* in the row $(*)'$ – just entries in that row. In fact, these are the sequences

- 1, 1, 1, ..., 1 (altogether k_0 sequences of lengths 3, 4, ..., $2 + k_0$),

- $1, 2, 3, \dots, 2 + k_0$ (1 time),
- $A_1^{+1}, A_1^{+1} + A_0^{+2}, A_1^{+1} + 2A_0^{+2}, \dots, A_1^{+1} + (1 + k_0)A_0^{+2}$ ($1 + k_2$ times),
- $A_2^{+1}, A_2^{+1} + A_1^{+2}, A_2^{+1} + 2A_1^{+2}, \dots, A_2^{+1} + (1 + k_0)A_1^{+2}$ ($1 + k_3$ times),
- $\dots \dots \dots \dots \dots$
- $A_s^{+1}, A_s^{+1} + A_{s-1}^{+2}, A_s^{+1} + 2A_{s-1}^{+2}, \dots, A_s^{+1} + (1 + k_0)A_{s-1}^{+2}$ ($1 + k_{s+1}$ times),
- $A_{s+1}^{+1}, A_{s+1}^{+1} + A_s^{+2}, A_{s+1}^{+1} + 2A_s^{+2}, \dots, A_{s+1}^{+1} + (1 + k_0)A_s^{+2}$ ($l_{s+2} - 1$ times)

(except for the irregular sequences in the beginning, the arithmetic sequences here have length $2 + k_0$, and their first (resp., second) terms are always in the row $(**')$ (resp., $(** *')$).

Summing up, the vector $\text{der}(\mathbf{WSST}_{k_0})$ is non-decreasing and consists of

- $2 + k_0$ times $1 = A_0$, then
- 1 time $2 + k_0 = A_1$, then
- $1 + k_2$ times $A_1^{+1} + (1 + k_0)A_0^{+2}$, then
- $1 + k_3$ times $A_2^{+1} + (1 + k_0)A_1^{+2}$, then
- $\dots \dots \dots \dots \dots$
- $1 + k_{s+1}$ times $A_s^{+1} + (1 + k_0)A_{s-1}^{+2}$, and eventually
- $l_{s+2} - 1$ times $A_{s+1}^{+1} + (1 + k_0)A_s^{+2}$.

To proceed, analogously as in the previous case $k_1 > 0$, we need

Lemma 2 *When $k_1 = 0$, then $A_j^{+1} + (1 + k_0)A_{j-1}^{+2} = A_{j+1}$ for $j = 1, 2, \dots, s, s + 1$.*

Attention. There is no use to directly compare this lemma with Lemma 1, because now we are under the restriction $k_1 = 0$. The two lemmas deal with *disjoint* families of integer sequences A .

Proof of lemma is inductive on j . For $j = 1$ it is quick, $A_1^{+1} + (1 + k_0)A_0^{+2} = 2 + k_1 + 1 + k_0 = 1 + 2 + k_0 = A_0 + A_1 = A_0 + (1 + k_1)A_1 = A_2$.

For $j = 2$ it is a bit longer,

$$\begin{aligned}
A_2^{+1} + (1 + k_0)A_1^{+2} &= 1 + (1 + k_2)A_1^{+1} + (1 + k_0)(2 + k_2) \\
&= 1 + 2(1 + k_2) + (1 + k_0)(1 + 1 + k_2) = 2 + k_0 + (1 + k_2)(1 + 2 + k_0) \\
&= A_1 + (1 + k_2)(A_0 + A_1) = A_1 + (1 + k_2)(A_0 + (1 + k_1)A_1) \\
&= A_1 + (1 + k_2)A_2 = A_3.
\end{aligned}$$

As for the inductive step, again in the form $(j-1, j) \Rightarrow j+1$, with, this time, $j \in \{2, 3, \dots, s\}$, we will nearly mimick the proof of Lemma 1. We will again make use of identity (4) and of a similar identity

$$A_j^{+2} = A_{j-2}^{+2} + (1 + k_{j+1})A_{j-1}^{+2} \quad (6)$$

which follows from the definition of the A sequences as well.

Let us check now if lemma's statement holds for $j+1$:

$$\begin{aligned} A_{j+1}^{+1} + (1 + k_0)A_j^{+2} &\stackrel{\text{by (4) and (6)}}{=} A_{j-1}^{+1} + (1 + k_{j+1})A_j^{+1} + (1 + k_0)(A_{j-2}^{+2} + (1 + k_{j+1})A_{j-1}^{+2}) \\ &= A_{j-1}^{+1} + (1 + k_0)A_{j-2}^{+2} + (1 + k_{j+1})(A_j^{+1} + (1 + k_0)A_{j-1}^{+2}) \\ &\stackrel{\text{by ind. ass.}}{=} A_j + (1 + k_{j+1})A_{j+1} = A_{j+2}. \end{aligned}$$

Lemma 2 is now proved by induction.

We are about to finish the proof of the case $k_1 = 0$ in the theorem. Indeed, Lemma 2 simplifies the expressions for the entries in the vector $\text{der}(\mathbf{WSST}_{k_0})$, while the l_0 operations G that are still to be performed add l_0 entries 1 on the left. Hence the vector $\text{der}(\mathbf{WSST}_{k_0}G_{l_0})$, is non-decreasing and consists of

- $2 + k_0 + l_0$ times A_0 , then
- $1 + k_1 = 1$ time A_1 , then
- $1 + k_2$ times A_2 , then
- $1 + k_3$ times A_3 , then
- $\dots \dots \dots$
- $1 + k_{s+1}$ times A_{s+1} , and eventually
- $l_{s+2} - 1$ times A_{s+2} .

Case $k_1 = 0$ [in the inductive proof of Theorem 2] is finished.

Theorem 2 is proved.

3.3 Reformulations of Theorem 2.

In order to simplify our to-be-done work in proving Theorem 3 in Section 4, let us slightly reformulate Theorem 2, then draw a corollary from that reformulation, and then yet another, more far-reaching, corollary.

Observation 1 *In the notation from Theorem 2, for an admissible word with $s+1$ letters S and such that $l_1 = l_2 = \dots = l_s = 0$, the sequence of operations*

$$S, T^{k_s}, S, T^{k_{s-1}}, \dots, S, T^{k_1}, S, T^{k_0}, G^{l_0}$$

started on the pair of arguments $(\underbrace{1, 1, \dots, 1}_{l_{s+1}-1})$ and $(\underbrace{1, 1, \dots, 1}_{l_{s+1}})$ produces as the eventual result the vector described in the statement of Theorem 2.

This is just a reformulation of Theorem 2, compare the beginning of its proof.

Corollary 1 *The sequence of operations*

$$S, T^{k_{n-1}}, S, T^{k_{n-2}}, \dots, S, T^{k_1}, S, T^{k_0}, G^{l_0}$$

(n is a natural integer not exceeding s) started on the pair of vector arguments $(\underbrace{w_0, w_0, \dots, w_0}_{m_0})$ and $(1, \underbrace{w_0, w_0, \dots, w_0}_{m_0})$, where w_0 is a positive integer, produces in the result the vector with the entries, consecutively:

- $2 + k_0 + l_0$ times A_0 , then
- $1 + k_1$ times A_1 , then
- $\dots \dots \dots$
- $1 + k_{n-1}$ times A_{n-1} , and eventually
- m_0 times $A_n w_0$.

This corollary is only slightly more general than Observation 1 (instead of s there is $n - 1$ now, m_0 instead of $l_{s+1} - 1$, and w_0 instead of 1) and its method of proof is the same as that in Theorem 2.

Corollary 2 *The same sequence of operations as in Corollary 1, when started on the pair of longer vector arguments*

$$(\underbrace{w_0, w_0, \dots, w_0}_{m_0}, \underbrace{w_1, w_1, \dots, w_1}_{m_1}, \dots, \underbrace{w_N, w_N, \dots, w_N}_{m_N})$$

and

$$(1, \underbrace{w_0, w_0, \dots, w_0}_{m_0}, \underbrace{w_1, w_1, \dots, w_1}_{m_1}, \dots, \underbrace{w_N, w_N, \dots, w_N}_{m_N}),$$

where w_0, w_1, \dots, w_N are positive integers, produces in the result the vector with the entries, consecutively:

- $2 + k_0 + l_0$ times A_0 , then
- $1 + k_1$ times A_1 , then
- $\dots \dots \dots$
- $1 + k_{n-1}$ times A_{n-1} , then
- m_0 times $A_n w_0$, then
- m_1 times $A_n w_1$, then
- $\dots \dots \dots$

- m_N times $A_n w_N$.

This is a direct consequence of Corollary 1. Indeed, replacing in that corollary w_0 and m_0 by w_i and m_i , respectively, $i \in \{1, 2, \dots, N\}$, and then *concatenating* the vector results for $i = 1, 2, \dots, N$ with the initial vector result in Corollary 1, we get the statement in Corollary 2.

4 Proof of Theorem 3

The proof of Theorem 3 will be by induction on $q \in \{0, 1, \dots, s\}$ in all the pairs of integers (q, s) , $0 \leq q \leq s$. The size of $s \geq q$ will not play any role in the arguments.

Before doing this we want to formulate and prove a statement that would greatly simplify the inductive step in that announced induction. Note that the notation used in Proposition 1 below is fully compatible with that of Corollary 2. It is done purposefully – in order for the reader to see that it is basically Corollary 2 again, only given in other terms.

Proposition 1 *Let the code of a geometric class \mathcal{C} have the form*

$$\mathbf{W} \mathbf{G} \mathbf{S} \mathbf{T}_{k_{n-1}} \mathbf{S} \mathbf{T}_{k_{n-2}} \dots \mathbf{S} \mathbf{T}_{k_1} \mathbf{S} \mathbf{T}_{k_0} \mathbf{G}_{l_0},$$

\mathbf{W} – the code of an ancestor of \mathcal{C} , ending on S or T, whose derived vector $\text{der}(\mathbf{W})$ is non-decreasing and has the different values $1 = w_0 < w_1 < \dots < w_N$ appearing in it with multiplicities m_0, m_1, \dots, m_N , respectively. Then $\text{der}(\mathcal{C})$ is non-decreasing as well and consists of

- $2 + k_0 + l_0$ times A_0 , then
- $1 + k_1$ times A_1 , then
-
- $1 + k_{n-1}$ times A_{n-1} , then
- m_0 times $A_n (= A_n w_0)$, then
- m_1 times $A_n w_1$, then
-
- m_N times $A_n w_N$.

Proof of Proposition 1. It is a direct consequence of Corollary 2, because the derived vector of \mathbf{W} can be taken as the first vector argument in that corollary, upon which the derived vector of $\mathbf{W} \mathbf{G}$ becomes the second vector argument in that corollary. (The operation \mathbf{G} adds a single 1 on the left, while the particular value $w_0 = 1$ in the proposition does not interfere with this action.)

The beginning of induction for $q = 0$ in **Theorem 3** is already done – it is Theorem 2, proved already for all $s \geq 0$.

The inductive step ‘ $q - 1 \Rightarrow q$ ’ in Theorem 3 for any fixed $q \geq 1$ and arbitrary $s \geq q$, assuming the due information for $q - 1$ and all $s' \geq q - 1$.

Let us focus our attention on the one before last from the left (or last, if $l_0 = 0$) segment of G ’s in the word of \mathcal{C} and truncate this word, calling \mathbf{W} the result of truncation, *before* the last letter G in that segment. That is, take

$$\mathbf{W} = G_{l_{s+1}} S T_{k_s} \dots S T_{k_{n_1}} G_{l_{n_1}-1}$$

where $l_{n_1}-1 \geq 0$ (the value 0 is not excluded). Then the vector $\text{der}(\mathbf{W})$ is known from the inductive assumption, for there are only $q-1$ positive values among the parameters $l_{n_1+1}, l_{n_1+2}, \dots, l_s$ related to the word \mathbf{W} , and $s' = s - n_1 \geq q - 1$. (Indeed, from our construction, $s - n_1 \geq n_q - n_1 > n_{q-1} - n_1 > \dots > n_2 - n_1 \geq 1$.) So what is precisely $\text{der}(\mathbf{W})$?

The inductive assumption *formally* would have yielded a description in terms of the set of letters (as if a formal alphabet)

$$\{n_1, n_2, \dots, n_{q-1}, s - n_1, k_0, l_0, \dots, k_{s-n_1}, l_{s-n_1}, l_{s-n_1+1}\},$$

because of the relevant bare cardinalities related to the word \mathbf{W} . Yet, in the concrete case of \mathbf{W} , all *indices* in the parameters k and l are naturally raised by $+n_1$, ranging from n_1 to $s+1$ (and not from 0 to $s-n_1+1$). And the *distances* in indices are $n_2-n_1, n_3-n_1, \dots, n_q-n_1$ instead of $n_1-0, n_2-0, \dots, n_{q-1}-0$, respectively (the distances in indices in \mathbf{W} are from n_1 , not from 0).

Taking into account all these specific features, the data about \mathbf{W} which follow from the inductive assumption, and are about to be plugged into Proposition 1, read:

In the vector $\text{der}(\mathbf{W})$ there appear $s - n_1 + 2$ different values $1 = w_0 < w_1 < w_2 < \dots < w_{s-n_1+1}$, listed below in the growing order, in the following $(q-1) + 1 = q$ separate rows:

$$\begin{aligned} & A_0^{+n_1}, A_1^{+n_1}, \dots, A_{n_2-n_1-1}^{+n_1}; \\ & A_{n_2-n_1}^{+n_1} (A_0^{+n_2}, A_1^{+n_2}, \dots, A_{n_3-n_2-1}^{+n_2}); \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & \prod_{j=1}^{q-2} A_{n_{j+1}-n_j}^{+n_j} (A_0^{+n_{q-1}}, A_1^{+n_{q-1}}, \dots, A_{n_q-n_{q-1}-1}^{+n_{q-1}}); \\ & \prod_{j=1}^{q-1} A_{n_{j+1}-n_j}^{+n_j} (A_0^{+n_q}, A_1^{+n_q}, \dots, A_{s-n_q+1}^{+n_q}). \end{aligned}$$

Moreover, $\text{der}(\mathbf{W})$ is non-decreasing and the multiplicities of its above-listed different values are as follows.

- $w_0 = A_0^{+n_1}$ appears $m_0 = 2 + k_{n_1} + (l_{n_1} - 1) = 1 + k_{n_1} + l_{n_1}$ times in row (remember that the last group of G's in \mathbf{W} consists, by construction, of $l_{n_1} - 1$ letters);
- for $1 \leq j \leq s - n_1$, w_j appears $m_j = 1 + k_{n_1+j} + l_{n_1+j}$ times in row (that is, $1 + k_{n_1+j}$ times, when $j \notin \{n_2 - n_1, n_3 - n_1, \dots, n_q - n_1\}$);
- the last and biggest w_{s-n_1+1} appears $m_{s-n_1+1} = l_{s+1} - 1$ times in row.

We are about to apply Proposition 1 to the code of the class \mathcal{C} under the induction step procedure and to its truncation \mathbf{W} , for $n = n_1$, $N = s - n_1 + 1$, the different values w_i in $\text{der}(\mathbf{W})$ and their multiplicities m_i ($i = 0, 1, \dots, s - n_1 + 1$), as specified above.

So Proposition 1 says that the table of different values in $\text{der}(\mathcal{C})$ is being obtained by just multiplying by A_{n_1} all the entries in the table of q rows for $\text{der}(\mathbf{W})$ above *and* by inserting on top of them the single row $A_0, A_1, \dots, A_{n_1-1}$. That is, the table of different values showing up in the vector $\text{der}(\mathcal{C})$ is of the form

$$\begin{aligned}
& A_0, A_1, \dots, A_{n_1-1}; \\
& A_{n_1}(A_0^{+n_1}, A_1^{+n_1}, \dots, A_{n_2-n_1-1}^{+n_1}); \\
& A_{n_1}A_{n_2-n_1}^{+n_1}(A_0^{+n_2}, A_1^{+n_2}, \dots, A_{n_3-n_2-1}^{+n_2}); \\
& \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
& A_{n_1} \prod_{j=1}^{q-2} A_{n_{j+1}-n_j}^{+n_j} (A_0^{+n_{q-1}}, A_1^{+n_{q-1}}, \dots, A_{n_q-n_{q-1}-1}^{+n_{q-1}}); \\
& A_{n_1} \prod_{j=1}^{q-1} A_{n_{j+1}-n_j}^{+n_j} (A_0^{+n_q}, A_1^{+n_q}, \dots, A_{s-n_q+1}^{+n_q}),
\end{aligned}$$

precisely as expected. Moreover, also by Proposition 1, the vector $\text{der}(\mathcal{C})$ is non-decreasing and the multiplicities of its listed different values which emerge from Proposition 1 are identical (recall that $n = n_1$) with those in the statement of Theorem 3. Theorem's statements for q positive parameters among $\{l_1, l_2, \dots, l_s\}$ are now justified and the step of induction is completed.

Theorem 3 is now proved.

Afterword. In the contribution [4], among many an issue raised, there is a far-reaching Question 9.19, consisting of parts 1, 2, and 3.

Part 1, especially when understood *sensu largo* (is it possible to produce Jean's recurrences in pure [Goursat] Monster terms?), is a true challenge.

Part 2 is elementary and covered by the present work (which, as a matter of record, completes [6]).

Part 3 is an invitation to research, for the mapping Pc (*Puiseux characteristic* being associated to a geometric class = a word over $\{R, V, T\}$) is explicit in [4], while the mapping GW ([small] growth vector being associated to a geometric

class = a word over $\{G, S, T\}$) is recursive in [3], and explicit in [6] and in the present contribution.

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